Operator Spectral Statistics, Chaos, and Asymptotic Freeness in Quantum Many-Body Systems

Viktor Jahnke

Gwangju Institute of Science Technology - Korea

viktorjahnke@gist.ac.kr

Collaboration with: KY Kim, H. Camargo, Y. Fu and K. Pal

Quantum Gravity of Open Systems
(IFT UAM-CSIC)

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Introduction

Chaos has played an important role in recent developments in Gauge/Gravity Duality.

- The realization that black holes are maximally chaotic has facilitated the identification of simple, lower-dimensional holographic models, which have been important in many recent advances in the field.
- Additionally, the search for random matrix behavior in the spectral properties of holographic systems has provided important insights into quantum gravity, particularly regarding the role of Euclidean wormholes in the gravitational path integral.

We can think of chaos as a guiding principle to test our ideas about quantum gravity, especially in contexts involving black holes.

Different avatars of quantum chaotic behavior

While classical chaos is well understood, a precise characterization of quantum chaos remains elusive. Although there are various probes for investigating quantum chaotic behavior, such as:

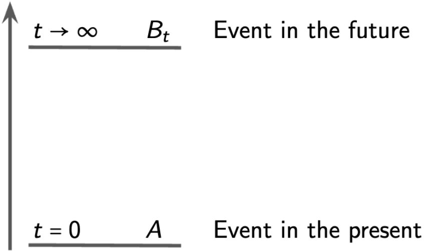
- Spectral Statistics
- Eigenstate Thermalization Hypothesis (ETH)
- Out-of-time-correlators (OTOCs)
- Krylov state and operator complexity

The connections between these probes are not entirely clear, and it remains uncertain whether there is a unifying framework that encompasses them all.

In this work, we investigate the connections between different notions of quantum chaos using **free probability theory**.

Chaos in Classical Dynamical Systems [Walters 1982, Gesteau 2023]

Classical Dynamical System = Probability Space + Time Evolution Map



Chaos makes events in the future statistically independent from events in the past.

This can be quantified by classical cumulants (connected correlation functions) involving A and B_t.

Probability Space and Independence

A **probability space** is a triple (Ω, Σ, μ):

- Ω: set of all possible states.
- Σ: σ-algebra of measurable subsets (events).
- $\mu : \Sigma \to [0,1]$: probability measure with $\mu(\Omega) = 1$.

Two events $A, B \in \Sigma$ are **independent** if:

$$\mu(A \cap B) = \mu(A)\mu(B)$$

Deviations from independence are measured by the cumulants:

$$F_2(A, B) = \mu(A \cap B) - \mu(A)\mu(B)\,.$$

Definition: A dynamical system consists of a probability space (Ω, Σ, μ) equipped with a time-evolution map $T_t : \Omega \to \Omega$, where $t \in \mathbb{R}$. We focus on maps that are **invertible** and **measure-preserving**: $\mu(A_t) = \mu(A)$ for any $A \subset \Omega$.

- Ω represents the phase space;
- Σ is a σ-algebra of measurable subsets of Ω;
- μ represents the state of the system (statistical ensemble).

Example: conservative system with N degrees of freedom with generalized coordinates and momenta $x = (q_1, \ldots, q_N, p_1, \ldots, p_N) \in \Omega$ and Hamiltonian $H(q_i, p_i)$.

- The orbit is confined on a constant-energy $(2N - 1)$-dimensional hypersurface Ω_E, defined by the condition:

$$E = H(q_i, p_i)\,.$$

- The measure μ on Ω reads:

$$d\mu = \prod_{i=1}^{N} dq_i\, dp_i\,.$$

  From μ one can compute the induced metric on Ω_E.

- Random variables are bounded functions in phase space, $f(q_i, p_i)$, and their expectation value is taken as follows:

$$\langle f \rangle = \int d\mu\, f(q_i, p_i)\,.$$

Classical ergodic hierarchy [Gesteau 2023, Ouseph et al 2023]

One can quantify chaos in classical dynamical systems by how quickly the system decorrelates events. This can be precisely quantified by cumulants , $F_2(A, B_t)$, whose vanishing indicates statistical independence between A and B_t.

A classical dynamical system is said to be:

- **Ergodic:** if $\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, F_2(A, B_t) = 0$, for all $A, B \subset \Sigma$.
- **Strong 2-Mixing:** if $\lim_{t\to\infty} F_2(A, B_t) = 0$, for all $A, B \subset \Sigma$.
- **K-mixing:** if $\lim_{t\to\infty} \sup_{B\in\Sigma_t} |F_2(A, B)| = 0$, for any subsets $A, \tilde{A} \subset \Sigma$, where Σ_t is the σ-algebra generated by $T_s(\tilde{A})$ for $s \geq t$,

Ergodic Hierarchy:

$$\boxed{\text{Ergodic} \Leftarrow \textbf{Strong Mixing} \Leftarrow \text{K-mixing} \Leftarrow \text{Anosov systems}}$$

Intuitive idea behind mixing

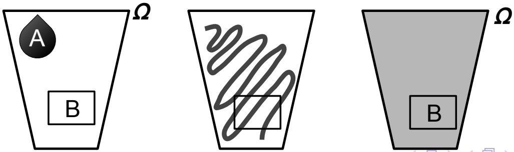
Mixing means that any subset $A \subset \Sigma$ is eventually spread uniformly across phase space Ω.

Example: Coffee dispersing in milk when stirred, leading to a uniform concentration (30% coffee, 70% milk)

- Concentration of coffee in the whole glass: $\frac{\mu(A)}{\mu(\Omega)} = 30\%$.
- Concentration of coffee in B at late times: $\frac{\mu(A_t \cap B)}{\mu(B)} = 30\%$.
- If the dynamics is mixing, the two ratios are the same, and we obtain:

$$\lim_{t\to\infty} F_2(A_t, B) = \lim_{t\to\infty} \mu(A_t \cap B) - \mu(A_t)\mu(B) = 0$$

where we use $\mu(\Omega) = 1$ and $\mu(A_t) = \mu(A)$.



Algebraic Perspective [Terrence Tao's blog 2010]

- In a classical probability space (Ω, Σ, μ), real-valued random variables are measurable functions $f : \Omega \to \mathbb{R}$.
- The set of such functions forms an algebra $\mathcal{A}$.

This algebraic structure allows us to abstract away the sample space and define the probability space algebraically.

$$(\Omega, \Sigma, \mu) \quad \longrightarrow \quad (\mathcal{A}, \varphi)\,,$$

where the functional $\varphi : \mathcal{A} \to \mathbb{R}$.

This algebraic perspective allows for the generalization of the definition of dynamical systems to the quantum case by allowing $\mathcal{A}$ to be a non-commutative algebra.

Quantum dynamical system [Gesteau 2023, Ouseph et al 2023]

A *quantum dynamical system* is a triple (M, ω, T_t), where:

- M is a von Neumann algebra of observables of the quantum system.
- $\omega : M \to \mathbb{C}$ is a normal state on M (quantum state);
- $T_t : M \to M$ is a one-parameter group of automorphisms of M, representing the time evolution of the system.

Mixing can be characterized using connected correlation functions of the form:

$$C(t) = \omega\big(A_t B\big) - \omega(A_t)\omega(B) = \langle A_t B\rangle - \langle A_t\rangle\langle B\rangle,$$

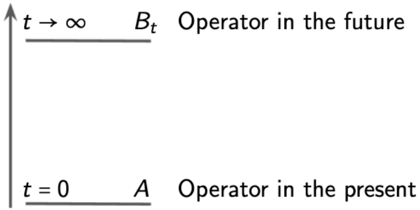
where $A_t = T_t(A)$.

Quantum 2-mixing is defined as follows: $\lim_{t\to\infty} C(t) = 0$ for any $A, B \in M$.

Connected correlation functions vanish at later times.

Chaos in Quantum Dynamical Systems

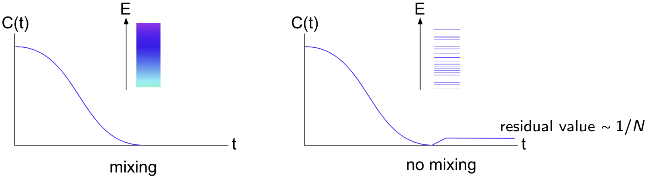
Quantum Dynamical System = Non-commutative Probability Space
+Time Evolution Map



Chaos makes events in the future statistically independent from events in the past.

This can be quantified by **free cumulants** (connected correlation functions) involving A and B_t.

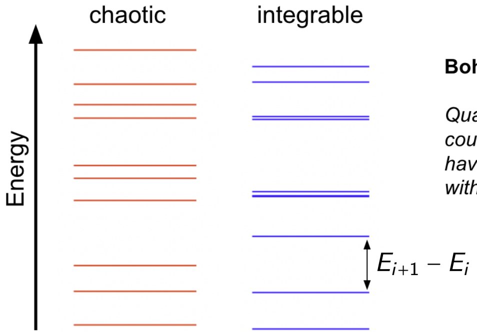
Mixing & Continuous vs Discrete Spectrum



- Quantum mixing requires the algebra of observables M to be of type III (continuous spectrum) [Ouseph et al 2023];
- Isolated quantum mechanical systems usually have a discrete energy spectrum (their algebra of observables is typically of type I);
- Therefore, isolated quantum mechanical systems do not exhibit quantum chaos in the strict sense of mixing, as their algebra of observables is typically of type I.

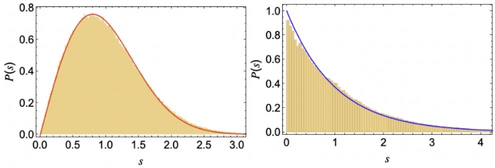
True quantum mixing can only occur in the thermodynamic limit, $N \to \infty$.

Quantum Signatures of Classical Chaos [Bohigas-Giannoni-Schmit 1984]



Bohigas-Giannoni-Schmit conjecture

Quantum systems whose classical counterparts exhibit chaotic behavior have energy level statistics that coincide with those of random matrix theory.



Level spacing statistics:

$$s_i = \rho(E_i)(E_{i+1} - E_i)$$

it should be done separately for each symmetry sector of the system

Chaos as Scrambling [Larkin, Ovchinnikov 1969; Shenker, Stanford 2013-2014; Kitaev 2014]

The scrambling of quantum information provides another notion of chaos.

Heisenberg picture:

$$W(t) = e^{iHt} W(0) e^{-iHt} = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} [H, [H, \cdots [H, W(0)] \cdots]]$$

Example: spin chain

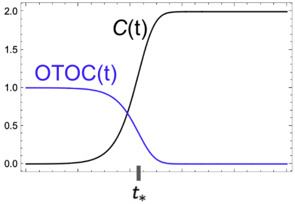
$$H = -\sum_i (Z_i Z_{i+1} + g X_i + h Z_i) \ ,$$

$$Z_1(t) = Z_1 - it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] + \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$\begin{aligned}
[H, Z_1] &\sim Y_1 \\
[H,[H,Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H,Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H,Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1X_2 + Y_1Y_2 + Z_1Z_2 + X_1Z_2 + \\
&\quad + Z_3Y_1 + Y_1Z_2Y_2 + Z_1X_2X_1 + X_2Z_3X_1
\end{aligned}$$

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta \underset{V^2 = W^2 = 1}{=} 2 - 2\, Re\left[\langle W(t)V(0)W(t)V(0)\rangle_\beta\right]$$

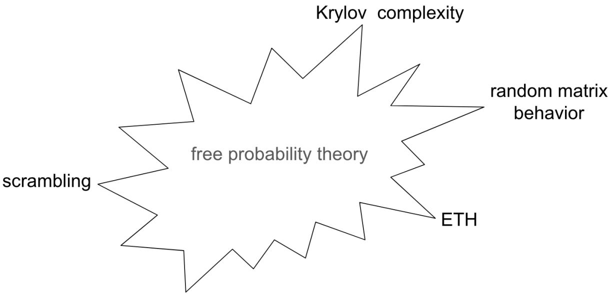
$$OTOC(t) = \langle W(t)V(0)W(t)V(0)\rangle_\beta$$



Scrambling of quantum information: OTOCs reach an equilibrium value near zero for almost any choice of operators V and W at some characteristic time scale t_*.

Free probability theory appears to offer the appropriate framework for mathematically precise connections between different notions of quantum chaos.

[Jindal, Hosur 2024 ;Chen, Kudler-Flam 2024]



In this work we use free probability theory to investigate the connection between **scrambling** and **operator spectral statistics**.

Free probability theory [Voiculescu, Speicher 2019]

Free probability theory extends classical probability theory to non-commuting random variables.

A non-commutative probability space is a defined by a pair $(\mathcal{A}, \varphi)$, where:

- $\mathcal{A}$ is an algebra of observables (typically a von Neumann algebra);
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a (positive and faithful) unital linear map.

Examples of non-commutative probability space:

- Quantum dynamical system $(\mathcal{A}, \varphi)$, with $\mathcal{A} \subset B(\mathcal{H})$, and $\varphi(a) = \langle \psi | a | \psi \rangle$;
- Algebra of $N \times N$ random matrices, with $\varphi(a) = \mathbb{E}\left[\frac{\mathrm{Tr}}{N}(a)\right]$.

$\mathbb{E}(\cdot)$ denotes the expectation value over some ensemble of random matrices.

Applications of free probability theory

Applications of free probability theory include:

- Determining the spectral distribution of sums and products of random matrices that exhibit asymptotic freeness [Voiculescu, Speicher];
- Formulation of generalizations of ETH [Pappalardi, Foini, Kurchan 2022];
- Derivation of the Page curve in evaporating black holes [Wang 2022];
- Characterization of quantum chaos [Jindal, Hosur 2024 ;Chen, Kudler-Flam 2024].

Freeness

In free probability theory, freeness is the non-commutative analogue of statistical independence in classical probability theory.

Two non-commutative random variables a and b of a given non-commutative probability space are free if

$$\langle (a-\langle a\rangle)^{p_1}(b-\langle b\rangle)^{q_1}\cdots(a-\langle a\rangle)^{p_n}(b-\langle b\rangle)^{q_n}\rangle = 0\,,$$

for all $n \geq 1$ for positive integer exponents p_m and q_m.

Freeness implies factorization:

$$\begin{aligned}
\langle a\,b\rangle &= \langle a\rangle\langle b\rangle\\
\langle a\,b\,a\rangle &= \langle a^2\rangle\langle b\rangle\\
\langle a\,b\,a\,b\rangle &= \langle a^2\rangle\langle b\rangle^2 + \langle a\rangle^2\langle b^2\rangle - \langle a\rangle^2\langle b\rangle^2
\end{aligned}$$

Random matrices and asymptotic freeness

In the context of random matrices, it is convenient to introduce the notion of *asymptotic freeness*.

Given a random matrix A, one defines φ as follows:

$$\varphi(A) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}(A)\right]$$

Notation: $\varphi(A) = \langle A \rangle$.

Two random matrices A and B are said to be asymptotically free is

$$\langle (A - \langle A \rangle)^{p_1} (B - \langle B \rangle)^{q_1} \cdots (A - \langle A \rangle)^{p_n} (B - \langle B \rangle)^{q_n} \rangle = 0\,,$$

for all $n \geq 1$ for positive integer exponents p_m and q_m.

Free additive convolution Voiculescu 1986

Given two self-adjoint random variables σ_1 and σ_2 with spectral distributions ρ_{σ_1} and ρ_{σ_2}, if σ_1 and σ_2 are *freely independent*, the spectral distribution of $\sigma_1 + \sigma_2$ is given by the free additive convolution of ρ_{σ_1} and ρ_{σ_2}, denoted as

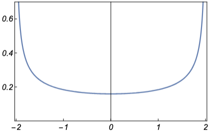
$$\rho_{\sigma_1+\sigma_2} = \rho_{\sigma_1} \boxplus \rho_{\sigma_2},$$

which can be computed using the R-transform.

Example 1: Sum of spin-1/2 operators

If $\rho_{\sigma_1} = \rho_{\sigma_2} = \frac{1}{2}(\delta_{-1} + \delta_1)$, and σ_1 and σ_2 are freely independent, then one can show that $\rho_{\sigma_1+\sigma_2}$ is given by an *arcsine distribution*.

$$\rho_{\sigma_1} = \tfrac{1}{2}(\delta_{-1} + \delta_1) \boxplus \rho_{\sigma_2} = \tfrac{1}{2}(\delta_{-1} + \delta_1) =$$

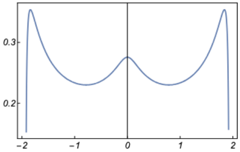


$\rho_{\sigma_1+\sigma_2} = \frac{1}{\pi\sqrt{4-\lambda^2}}, \ \lambda \in (-2, 2)$

Example 2: Sum of spin-1 operators

If $\rho_{\Sigma_1} = \rho_{\Sigma_2} = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$, and Σ_1 and Σ_2 are freely independent, then one can show that $\rho_{\Sigma_1+\Sigma_2}$ is given by the following distribution:

$$\rho_{\Sigma_1}\left(= \tfrac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)\right) \boxplus \rho_{\Sigma_2}\left(= \tfrac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)\right) =$$



$\rho_{\Sigma_1+\Sigma_2}$, $\lambda \in (-1.9226, 1.9226)$

$$\rho_{\Sigma_1+\Sigma_2}(\lambda) = \left| \frac{f_1^2(\lambda) - 4 - 9\lambda^4 + 33\lambda^2}{3\sqrt{3}\pi\lambda\,(\lambda^2 - 4)\, f_1(\lambda)} \right| ,$$

where

$$f_1(\lambda) = \left(8 + 9\left(3\lambda^4 - 30\lambda^2 + 70\right)\lambda^2 + 27\sqrt{-\lambda^2\left(\lambda^2 - 4\right)^2\left(9\lambda^4 - 33\lambda^2 - 1\right)}\right)^{1/3} .$$

Asymptotic Freeness & Chaos [Chen, Kudler-Flam 2024]

- Asymptotic freeness provides a new way to characterize chaos in quantum many-body systems.
- It is based on the asymptotic free independence of time-evolved operators with respect to operators at $t = 0$.

Key Idea: Consider two operators A and B in a quantum system. If the dynamics is chaotic, the time-evolved operator

$$B(t) = e^{iHt} B \, e^{-iHt}$$

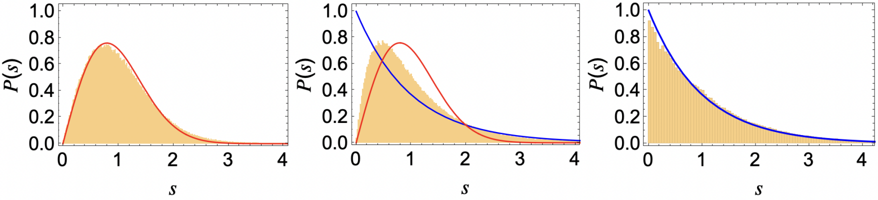
will be asymptotically free from A.

Test for Freeness: If the eigenvalues of $A + B(t)$ follow the predictions of free probability theory, this signals asymptotic freeness.

In this work, we study the emergence of the arcsine law in quantum many-body systems across an integrable to chaotic transition.

$$H = -\sum_{i=1}^{L-1} Z_i Z_{i+1} - \sum_{i=1}^{L} (h_x X_i + h_z Z_i) + g \sum_{i=1}^{L} \epsilon_i X_i \,,$$

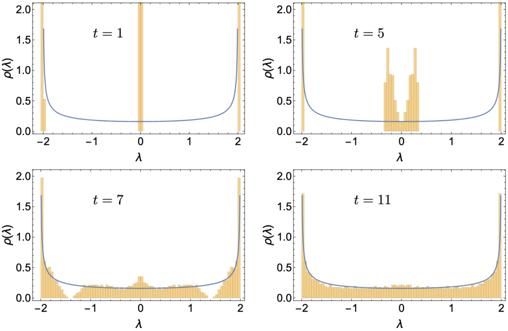
where $\epsilon_i \sim N(0,1)$, open boundary conditions.



We fix $(h_x, g) = (-1, 0.2)$ and vary $h_z = 0.16$ (left), 0.05 (center), 0 (right).

Emergence of the arcsine law

Density of eigenvalues of $Z_1(0) + Z_{10}(t)$ at the chaotic point for $L = 10$ sites.

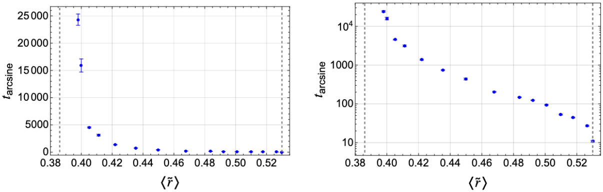


Similar results are observed regardless of the choice of the Hamiltonian parameters, provided the dynamics is not integrable.

Arcsine time vs r-parameter statistics

We fix $(h_x, g) = (-1, 0.2)$ and vary h_z from 0.16 (chaotic point) to a small value close to the integrable point.

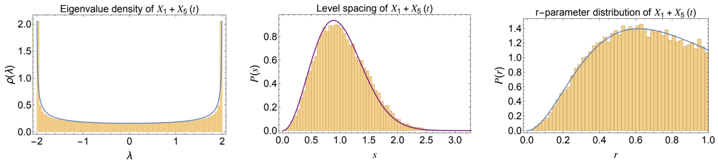
Correspondingly, the r-parameter varies from 0.53 (GOE) to 0.38 (Poisson).



The arcsine distribution is reached at increasingly later times as one approaches the integrable point.

Fluctuations on top of the arcsine distribution

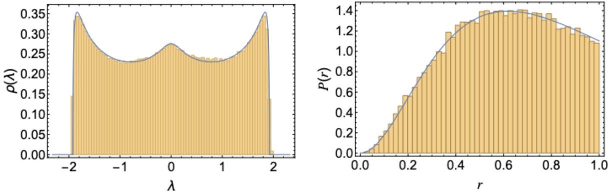
The fluctuations on top of the arcsine distribution seen to follow a universal Wigner-Dyson distribution even when the dynamics is close to the integrable point.



This suggests a connection between level spacing statistics of operators and asymptotic freeness.

We observe similar results for the SYK model, and for the mixed-field Ising model without disorder.

Spectral statistics of the operator $\sqrt{2/3}(Y_1^{(s)} + Y_6^{(s)}(t))$ based on 100 realizations of a spin-1 generalization of the mixed-field Ising model.



(a) Eigenvalue density (b) r-parameter distribution

The fluctuations are well described by a GUE of random matrices.

Conclusions

- We argue that free probability theory offers a precise mathematical framework for characterizing quantum chaos, where mixing is effectively replaced by asymptotic freeness. Moreover, it connects different notions of quantum chaos.
- Employing free probability theory, we present evidence that the spectral statistics of operators in quantum many-body systems serve as a robust diagnostic for quantum chaotic behavior.
- We demonstrate that the level spacing statistics of sums of asymptotically free operators exhibit universal level spacing repulsion.

Future Directions

- Employ free probability theory to explore the emergence of **BPS chaos**, a random matrix behavior observed in the ground state of certain holographic systems.
- Investigate whether the random matrix behavior observed in strongly chaotic quantum mechanical systems without a classical limit can be explained through free probability theory.
- Investigate the connection between asymptotic freeness and K-mixing. Can free probability theory provide insight into the early-time exponential behavior of OTOCs observed in certain systems?

THE END

Cauchy Transform and Spectral Distribution

For an operator a with spectral distribution $\rho(\lambda)$ compactly supported on $\mathbb{R}$, the Cauchy transform is defined as

$$G(z) = \int_{\mathbb{R}} d\lambda \frac{\rho(\lambda)}{z - \lambda}.$$

Using the Stieltjes inversion formula, the spectral density is obtained from $G(z)$ via

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \operatorname{Im} G(\lambda + i\epsilon).$$

R-**Transform:** The inverse of $G(z)$, denoted $\mathcal{B}(z)$, has the form

$$G^{-1}(z) = \mathcal{B}(z) = \frac{1}{z} + R(z),$$

where $R(z)$ is the R-transform of $\rho(\lambda)$.

Free Cumulants and Free Convolution

Free Cumulants: The R-transform encodes free cumulants:

$$R(z) = \sum_n k_n(a) z^{n-1},$$

where $k_n(a)$ satisfies

$$k_n(a_1 + \cdots + a_N) = k_n(a_1) + \cdots + k_n(a_N).$$

Free Convolution: For free random variables $a_1, \ldots, a_N$,

$$R_{\rho_{a_1} \boxplus \cdots \boxplus \rho_{a_N}}(z) = R_{\rho_{a_1}}(z) + \cdots + R_{\rho_{a_N}}(z).$$

This provides a method to compute the spectral distribution of sums of free operators.

Example: Sum of Spin-1/2 Operators

Consider the sum of two free operators σ_1 and σ_2, whose spectral distribution follows a Bernoulli distribution:

$$\rho_\sigma(\lambda) = \frac{\delta(\lambda - 1) + \delta(\lambda + 1)}{2}.$$

Step 1: Compute the Cauchy Transform

$$G_\sigma(z) = \frac{z}{z^2 - 1}.$$

Step 2: Compute the R-Transform Using $G^{-1}(z) = \frac{1}{z} + R(z)$, we find:

$$R_\sigma(z) = \frac{\sqrt{1 + 4z^2} - 1}{2z}.$$

Example: Sum of Spin-1/2 Operators (Cont.)

Step 3: Apply Free Convolution For two asymptotically free Pauli operators:

$$R_{\sigma_1+\sigma_2}(z) = \frac{\sqrt{1+4z^2}-1}{z}.$$

Step 4: Compute $G(z)$ Solving for $G(z)$, we obtain:

$$G_{\sigma_1+\sigma_2}(z) = \frac{1}{\sqrt{z^2-4}}.$$

Step 5: Compute $\rho(\lambda)$ Using the Stieltjes inversion formula:

$$\rho_{\sigma_1+\sigma_2}(\lambda) = \frac{1}{\pi\sqrt{4-\lambda^2}}, \quad \text{for } |\lambda| < 2,$$

vanishing for $|\lambda| > 2$, which is the arcsine law.

Higher spin Ising model

We consider a spin-s generalization of the mixed-field Ising model, where the spin-$\frac{1}{2}$ particles at each site are replaced by spin-s representations of $SU(2)$. We start with the model:

$$H^{(s)} = -2\frac{\sqrt{s(s+1)}}{\sqrt{3}}\left[\sum_{i=1}^{L-1} Z_i^{(s)} Z_{i+1}^{(s)} + \sum_{i=1}^{L}\left(h_x X_i^{(s)} + h_z Z_i^{(s)}\right) + g\sum_{i=1}^{L}\epsilon_i X_i^{(s)}\right],$$

where $X_i^{(s)}$, $Y_i^{(s)}$, and $Z_i^{(s)}$ are the matrices corresponding to the spin-s representation of $SU(2)$ and ϵ_i is drawn from a Gaussian distribution with average zero and unit variance.